Derivation of the predictive distribution with unknonwn mean and known variance

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1 Normal prior

Assume the following:

$$Y \sim N(\theta, 1) \qquad \theta \sim N(0, c^{-1}) \tag{1}$$

The predictive distribution of Y given the observation y_1 is obtained as follows:

$$p(Y|y_1) = \int p(Y|y_1, \theta)p(\theta|y_1)d\theta \tag{2}$$

To obtain the posterior, I first multiply the likelihood by the prior.

$$p(\theta)p(y_1|\theta) = \phi(\theta|0, c^{-1})\phi(y_1|\theta, 1)$$

Next note that

$$\begin{aligned} c\theta^2 + (y_1 - \theta)^2 &= c\theta^2 + y_1^2 - 2y_1\theta + \theta^2 \\ &= (1+c)\theta^2 - 2y_1\theta + y_1^2 \\ &= (1+c)[\theta^2 - 2\frac{y_1}{1+c}\theta + (\frac{y_1}{1+c})^2)] - \frac{y_1^2}{1+c} + y_1^2 \\ &= (1+c)(\theta - \frac{y_1}{1+c})^2 + \frac{c}{1+c}y_1^2 \end{aligned}$$

and therefore:

$$p(\theta)\phi(y_1|\theta,1) = \bar{c}\phi(\theta|(1+c)^{-1}y_1,(1+c)^{-1}) \tag{3}$$

where $\bar{c} \equiv \phi(y_1|0,(1+c)/c)$ is the marginal distribution of Y and the other term is the posterior. Substituting in (2), the predictive density of Y is given by:

$$p(Y|y_1) = \int \phi(Y|\theta,1)\phi(\theta|(1+c)^{-1}y_1,(1+c)^{-1})d\theta \tag{4}$$

Note that:

$$\begin{aligned}
(Y-\theta)^2 + (1+c)[\theta - (1+c)^{-1}y_1]^2 = \\
= Y^2 - 2Y\theta + \theta^2 + (1+c)[\theta^2 - 2(1+c)^{-1}y_1\theta + (1+c)^{-2}y_1^2] = \\
= (2+c)\theta^2 - 2(Y+y_1)\theta + Y^2 + (1+c)^{-1}y_1^2 = \\
= (2+c)[\theta - (2+c)^{-1}(Y+y_1)]^2 - (2+c)^1(Y+y_1)^2 + Y^2 + (1+c)^{-1}y_1^2 = \\
= (2+c)[\theta - (2+c)^{-1}(Y+y_1)]^2 + (2+c)^{-1}(1+c)[Y-(1+c)^{-1}y_1]^2
\end{aligned}$$

Integrating out θ gives:

$$Y|y_1 \sim N((1+c)^{-1}y_1,(1+c)^{-1}(2+c)) \tag{5}$$

2 Laplace prior

Consider $Y \sim N(\theta,1)$ with Laplace prior $p(\theta;c) = c/2\exp(-c|\theta|)$. To compute the mean and variance of the predictive density, notice that $Y = (Y-\theta)+\theta$ and

$$\begin{aligned}
E(Y|y) &= E[(Y-\theta)+\theta|y] \\
&= E[E(Y-\theta|\theta)|y] + E(\theta|y) \\
&= E(\theta|y) \\
V(Y|y) &= E(Y^2|y) - E(Y|y)^2 \\
&= E[(Y-\theta+\theta)^2|y] - E(\theta|y)^2 \\
&= E[E[(Y-\theta)^2 + 2\theta(Y-\theta) + \theta^2|\theta]|y] - E(\theta|y)^2 \\
&= 1 + E(\theta^2|y) - E(\theta|y)^2
\end{aligned} \tag{6}$$

Therefore, to find the mean and variance of the predictive density, it is sufficient to find the mean and variance of the posterior. The posterior is given by:

$$p(\theta|y) = p(\theta)p(y|\theta)/p(y) \tag{7}$$

Let's start by computing the numerator of the posterior:

$$p(\theta)p(y|\theta) = c/2\exp\{-c|\theta|\}(2\pi)^{-1/2}\exp\{-1/2(y-\theta)^2\}$$

Consider the argument of the exponential first:

$$\begin{aligned} * &= -1/2I(\theta<0)[-2c\theta+y^2-2y\theta+\theta^2]+ \\ &\quad -1/2I(\theta>0)[2c\theta+y^2-2y\theta+\theta^2] \\ &= -1/2I(\theta<0)[(\theta-(y+c))^2-2cy-c^2] \\ &\quad -1/2I(\theta>0)[(\theta-(y-c))^2+2cy-c^2] \end{aligned} \tag{8}$$

Applying the properties of the truncated normal distributions, the marginal distribution of y is:

$$\begin{aligned} p(y) &= \int_{-\infty}^{\infty} p(\theta)p(y|\theta)d\theta \\ &= c/2[\exp\{-1/2(-2cy-c)^2\}\int_{-\infty}^{0}(2\pi)^{-1/2}\exp\{-1/2[\theta-(y+c)]^2\}d\theta\}+ \\ &\quad +\exp\{-1/2(2cy-c)^2\}\int_{0}^{\infty}(2\pi)^{-1/2}\exp\{-1/2[\theta-(y-c)]^2\}d\theta\}]= \\ &= c/2[\exp\{-1/2(-2cy-c)^2\}\Phi(-y-c)+\exp\{-1/2(2cy-c)^2\}[1-\Phi(-y+c)]= \\ &= c/2[\exp\{-1/2(-2cy-c)^2\}\Phi(-y-c)+\exp\{-1/2(2cy-c)^2\}\Phi(y-c)]= \\ &= c/2\exp(c^2/2)[\exp(cy)\Phi(-y-c)+\exp(-cy)\Phi(y-c)] \\ &\equiv c/2\exp(c^2/2)K \end{aligned} \tag{9}$$

The mean of the posterior is:

$$
\begin{aligned}
E(\theta|y) = &\int_{-\infty}^{\infty} \theta p(\theta|y) = \\
&= K^{-1}[\exp(cy)\Phi(-y-c)\int_{-\infty}^{0} \frac{(2\pi)^{-1/2}}{\Phi(-y-c)}\theta \exp\{-1/2[\theta-(y+c)]^2\}d\theta + \\
&\quad + \exp(-cy)\int_{-\infty}^{\infty}(2\pi)^{-1/2}\theta \exp\{-1/2[\theta-(y-c)]^2\}d\theta + \\
&\quad - \exp(-cy)\Phi(-y+c)\int_{-\infty}^{0} \frac{(2\pi)^{-1/2}}{\Phi(-y+c)}\theta \exp\{-1/2[\theta-(y-c)]^2\}d\theta] \\
&= K^{-1}[\exp(cy)\Phi(-y-c)(y+c-\frac{\phi(-y-c)}{\Phi(-y-c)}) + \\
&\quad + \exp(-cy)(y-c) + \\
&\quad - \exp(-cy)\Phi(-y+c)(y-c-\frac{\phi(-y+c)}{\Phi(-y+c)})] \\
&= K^{-1}[\exp(cy)\Phi(-y-c)+\exp(-cy)(1-\Phi(-y+c))]y + \\
&\quad - K^{-1}[-\exp(cy)\Phi(-y-c)+\exp(-cy)(1-\Phi(-y+c))]c + \\
&\quad - K^{-1}[\exp(cy)\phi(-y-c)-\exp(-cy)\phi(-y+c)] \\
&= y - K^{-1}[-\exp(cy)\Phi(-y-c)+\exp(-cy)\Phi(y-c)]c \\
&= y - \frac{1-\exp(2cy)\frac{\Phi(-y-c)}{\Phi(y-c)}}{1+\exp(2cy)\frac{\Phi(-y-c)}{\Phi(y-c)}} \cdot c \qquad (10)
\end{aligned}
$$

The final element needed to compute the variance of the posterior is:

$$
\begin{aligned}
E(\theta^2|y) = &\int_{-\infty}^{\infty} \theta^2 p(\theta|y) = \\
&= K^{-1}[\exp(cy)\Phi(-y-c)\int_{-\infty}^{0} \frac{(2\pi)^{-1/2}}{\Phi(-y-c)}\theta^2 \exp\{-1/2[\theta-(y+c)]^2\}d\theta + \\
&\quad + \exp(-cy)\int_{-\infty}^{\infty}(2\pi)^{-1/2}\theta^2 \exp\{-1/2[\theta-(y-c)]^2\}d\theta + \\
&\quad - \exp(-cy)\Phi(-y+c)\int_{-\infty}^{0} \frac{(2\pi)^{-1/2}}{\Phi(-y+c)}\theta^2 \exp\{-1/2[\theta-(y-c)]^2\}d\theta] \\
&= K^{-1}[\exp(cy)\Phi(-y-c)(E_1^2+\tilde{V}_1) + \\
&\quad + \exp(-cy)[1+(y-c)^2] + \\
&\quad - \exp(-cy)\Phi(-y+c)(E_2^2+\tilde{V}_2)]
\end{aligned}
$$

where $E_i = -b_i - \frac{\phi(b_i)}{\Phi(b_i)}$ and $\tilde{V}_i = 1 - b_i \frac{\phi(b_i)}{\Phi(b_i)} - [\frac{\phi(b_i)}{\Phi(b_i)}]^2$, $i = \{1, 2\}$, $b_1 = -y - c$ and $b_2 = -y + c$ are the mean and variance of a truncated normal distribution. Noting that $E_i^2 + V_i = 1 + b_i \frac{\phi(b_i)}{\Phi(b_i)} + b_i^2$, the expression can be further simplified as follows.

$$
\begin{aligned}
E(\theta^2|y) = {} & K^{-1}[\exp(cy)\Phi(-y-c) + \exp(-cy)(1 - \Phi(-y+c))] + \\
& + K^{-1}[\exp(cy)\phi(-y-c)(-y-c) - \exp(-cy)\phi(-y+c)(-y+c)] + \\
& + K^{-1}[\exp(cy)\Phi(-y-c)(-y-c)^2 + \exp(-cy)(1 - \Phi(-y+c))(-y-c)^2] = \\
= {} & 1 - \frac{2\frac{\phi(y-c)}{\Phi(y-c)} \cdot c - [(-y+c)^2 + \exp(2cy)\frac{\Phi(-y-c)}{\Phi(y-c)}(-y-c)^2]}{1 + \exp(2cy)\frac{\Phi(-y-c)}{\Phi(y-c)}} \qquad (11)
\end{aligned}
$$